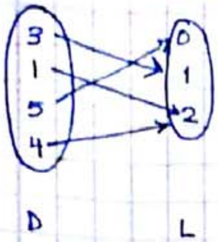
eg

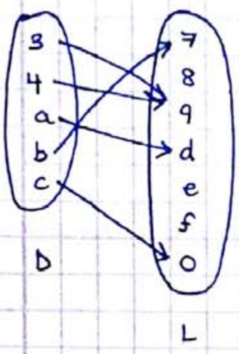


this is a function.

∴ Functions can be:
- one – to – one
- many – to – one

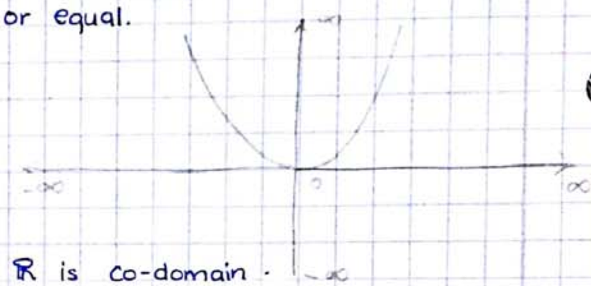
Functions cannot be:
- one-to-many

eg



is a function.
Domain = D
Co-domain = L
Range = {7, 9, d, 0} = Image

Range $\subseteq$ co-domain.
↳ subset or equal.



Function representation:

$f : \mathbb{R} \longrightarrow \mathbb{R}$
$f(x) = x^2$

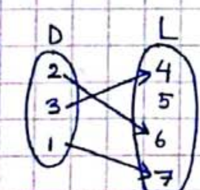
where the first $\mathbb{R}$ is the domain and second $\mathbb{R}$ is co-domain.
x-axis is domain and y axis is co-domain. Range from $0 \longrightarrow \infty$.
Here, Range $\neq$ co-domain (no negative f value is an image from an element in domain)
Range $[0, \infty)$. Range $\subseteq$ co-domain.

Definition: Assume $f : D \longrightarrow L$ is a function. We say f is onto (surjective) if co-domain = Range.

eg: $f : \mathbb{R} \longrightarrow [0, \infty)$
$f(x) = x^2$ is now onto. bc codomain = range
(injective)

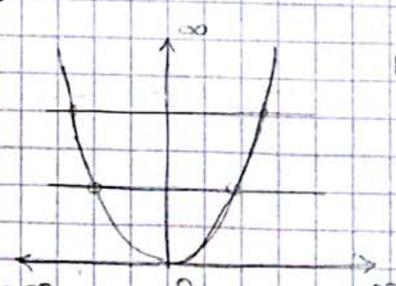
We say f is 1-1 if:
- 2 differents elements in the domain correspond to 2 different elements in the co-domain.
- ~~For~~ each element in the range corresponds to one and only one element in the domain.
- whenever $f(a_1) = f(a_2)$, for some $a_1, a_2 \in$ Domain, then $a_1 = a_2$

eg:



this is a function.
function is one to one.
function is not onto.

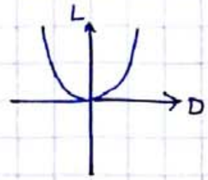
Domain = {2, 3, 1}
Co-domain = {4, 5, 6, 7}
Range = {4, 6, 7}



horizontal line check for 1-1

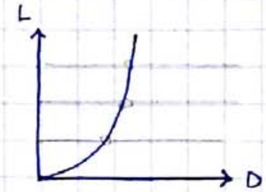
eg: $f : \mathbb{R} \rightarrow [0, \infty)$
$f(x) = x^2$ is onto
is not one to one

Def: $f: D \to L$ is called a <u>bijective</u> function if it is 1-1 AND onto



eg $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$.

- not onto Range $[0,\infty) \neq \mathbb{R}$
- not 1-1



eg: $f: [0,\infty) \to [0,\infty)$
$f(x) = x^2$

- is onto
- is one-to-one
- is <u>bijective</u>

Def: Assume $f: D \to L$ is a <u>bijective</u> function.
then $f^{-1}: L \to D$ is a function that is also a <u>bijective</u> function.

note: f^{-1} <u>does not</u> mean $\frac{1}{f}$

Suppose two functions:

$\boxed{1}$ $\left.\begin{array}{l} f_1 = x^2 \\ f_2 = x+1 \end{array}\right\}$ Assume $f: \mathbb{R} \to \mathbb{R}$

composition

$f_1 \circ f_2 = f_1(f_2(x)) = f_1(x+1) = (x+1)^2 = x^2 + 2x + 1.$

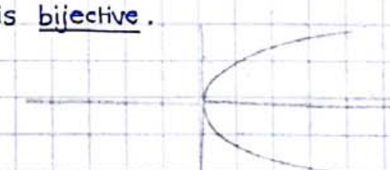
$f_2 \circ f_1 = f_2(f_1(x)) = f_2(x^2) = x^2 + 1.$

Observe: the composition does not ~~compute~~ <u>commute</u> i.e $f_1 \circ f_2 \neq f_2 \circ f_1$
so <u>order matters</u>..

if $f_1: \mathbb{R} \to \mathbb{R}$
$f_1(x) = x^2$ does not have an inverse not bijective

but $f_1: [0,\infty) \to [0,\infty)$
$f_1(x) = x^2$ has an inverse bc f_1 is <u>bijective</u>.

$f_1^{-1} = \underset{L}{[0,\infty)} \to \underset{D}{[0,\infty)}$



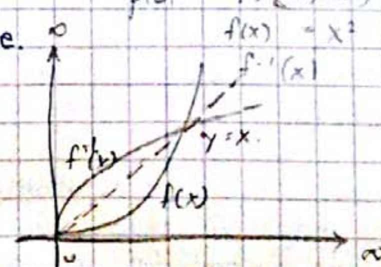
$y = f_1(x) = x^2$
swap: $x = y^2$
solve for y: $y = \pm\sqrt{x}$, we choose $+\sqrt{x}$ since Range $[0,\infty]$
$\therefore f^{-1}(x) = \sqrt{x}$

now ~~S~~ $f_1 \circ f_1^{-1} = f_1(\sqrt{x}) = x$
So, $f_1(x)$ and $f_1^{-1}(x)$ are symmetric <u>along the line $y = x$</u>

Note. $x^2 + y^2 = 4$ is <u>not</u> a function.
∴ cannot describe it as onto, 1-1 or bijective.

---.

plot $f: [0,\infty) \to [0,\infty)$
$f(x) = x^2$
$f^{-1}(x)$

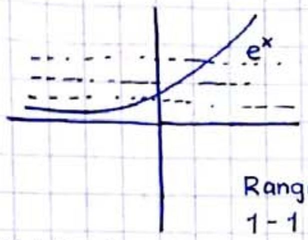


BRUNNEN

eg: $f: \mathbb{R} \to (0, \infty)$ 0 not included.
$f(x) = e^x$

is $f(x)$ bijective?
if yes, find $f^{-1}(x)$.

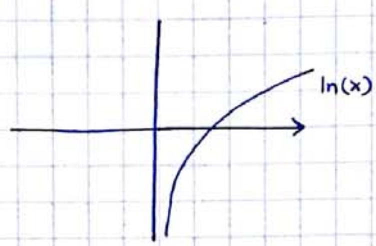
Ans: Draw.



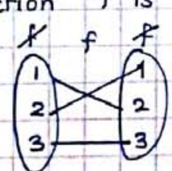
Range $(0, \infty)$ = Codomain. ✓
1-1 ✓

yes $f(x)$ is bijective.

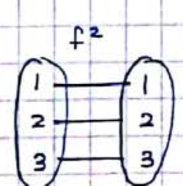
Finding $f^{-1}(x)$
$f: (0, \infty) \to \mathbb{R}$
$y = f(x) = e^x$
$x = f^{-1}(y) = e^y$
solve for y: $y = \ln(x)$
$f^{-1}(x) = \ln(x)$



Suppose function f is 1-1 and onto (~~bijection~~ bijective)

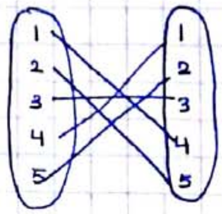


$f^2 = f \circ f$
then $(f \circ f)(1) = 1$
$(f \circ f)(2) = 2$
$(f \circ f)(3) = 3$



identity map $= I$ (i.e $y = x$)
Identity map will map each element to itself
Identity map composite any other bijective function will result in same bijective function

eg: any ~~a~~ bijective function with finite set can be represented as cycle



→ $f = (1\ 4)(2\ 5)$ (3 is not mentioned bc it maps to itself).

Ques: Find least integer value n s.t $f^n = I$

Ans: $n =$ LCM(~~|set 1|, |set 2|~~) LCM (|cycle 1|, |cycle 2|)

$= \text{LCM}(2, 2) = 2$

$\therefore\ f^2 = I,\ f \circ f = I.$

Eg: $f = (1\ 2\ 3)(4\ 6\ 5)(7\ 8)$

Domain:	1	2	3	4	5	6	7	8
Co-domain = Range:	2	3	1	6	4	5	8	7

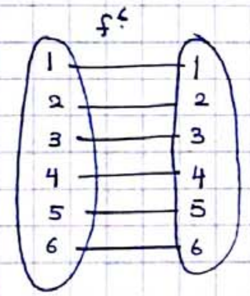
Find least integer value s.t $f^n = I$

$n = \text{LCM}(3, 3, 2) = 6$ i.e $f^6 = I$

check.

$f^2 =$	3	1	2	5	6	4	7	8
$f^3 =$	1	2	3	4	5	6	8	7
$f^4 =$	2	3	1	6	4	5	7	8
$f^5 =$	3	1	2	5	6	4	8	7
$f^6 =$	1	2	3	4	5	6	7	8

f^6



Identity Map

LCM [6, 8, 12, 14] :

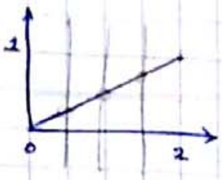
2 | 6, 8, 12, 14
2 | 3, 4, 6, 7
2 | 3, 2, 3, 7
3 | 3, 1, 3, 7
7 | 1, 1, 1, 7
1, 1, 1, 1

$\therefore \text{LCM} = 2^3 \times 3 \times 7$

Homework :

[I] (i) Let $f : (0, 2) \to (0, 1]$ s.t. $f(x) = 0.5x$. ~~If~~ Is f a function? Is it 1-1? Is it onto? Explain briefly.

ans :



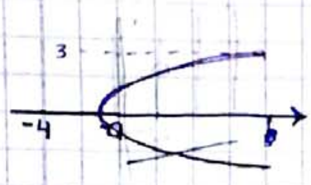
vertical line check

f is a function. It is 1-1 bc. every value in the co-domain is mapped back to exactly one element in the domain. f is not onto bc range $\neq$ co domain

The domain is almost 2 → image will be almost 1 but 1 is not mapped to anything in domain.

ii) let $f : (-4, 8) \to (0, 3)$ s.t $f(x) = \sqrt{x+1}$. is f a function? is it injective? is it surjective? Explain.

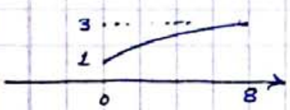
ans.



f is not a function bc for values $x < -1$, there is no corresponding value in the codomain ($x < -1$ has no image). And the elements in the domain maps to more than one element in the codomain (one to many)

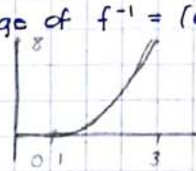
iii) Let $f : (0,8) \to (a\ b)$ s.t $f(x) = \sqrt{x+1}$. Find a, b so that f is bijective. Then find domain and range of f^{-1}. Write down the eq^n of f^{-1}.

ans



in order for f to be ~~bijectio~~ bijective $a = 1$ and $b = 3$.

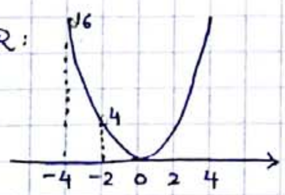
domain of $f^{-1} = (1, 3)$.
range of $f^{-1} = (0, 8)$.



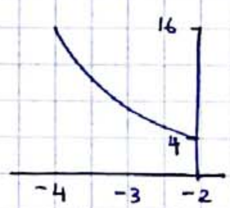
$f^{-1} \Rightarrow x = \sqrt{y+1}$
$x^2 - 1 = y$
$\therefore f^{-1}(x) = x^2 - 1$

iv) Let $f :$ ~~R~~ $(-4, b) \to (a, 4)$ s.t $f(x) = x^2$. Find a, b so that f is bijective. Then find the domain and range of f^{-1}. Write down the eqn for f^{-1}.

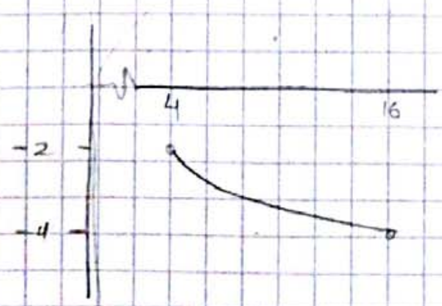
ans : Assuming $\mathbb{R} \to \mathbb{R}$:



To make f bijective, $a = 16$, $b = -2$:



Domain of f^{-1} : $(16, 4)$
Range of f^{-1} : $(-4, -2)$



f^{-1} : $y = x^2$
$x = y^2$
$y = \mp\sqrt{x}$

$\therefore f^{-1}(x) = -\sqrt{x}$

v) Let $f : \{1\ 2\ 3\ 4\ 5\ 6\ 7\} \to \{1\ 2\ 3\ 4\ 5\ 6\ 7\}$ s.t
$f(1) = 2$, $f(2) = 1$, $f(3) = 4$, $f(4) = 5$, $f(5) = 6$, $f(6) = 7$ and $f(7) = 3$

i.e. $f ==$

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 4 & 5 & 6 & 7 & 3 \end{matrix}$$

Find f^2 and f^3. Write f as a composition of disjoint cycles, then find the smallest possible integer $n \geq 1$ s.t. $f^n = I$

ans: F as disjoint ~~set~~ cycles: $(1\ 2)(3\ 4\ 5\ 6\ 7)$

$f^2 = 1 \quad 2 \quad 5 \quad 6 \quad 7 \quad 3 \quad 4$

$f^3 = 2 \quad 1 \quad 6 \quad 7 \quad 3 \quad 4 \quad 5$

$n = LCM(2,5) = 10$ so $f^{10} = I$

2: Assume F_1 ...

Statement: $\mathbb{Q}$ is Countable. Hence $|\mathbb{Q}| = |\mathbb{Z}| = |\mathbb{N}|$

Proof: $F_1 = \mathbb{Z}$, which is Countable

$F_2 = \frac{1}{2} + \mathbb{Z} = \{\frac{1}{2} + a \mid a \in \mathbb{Z}\}$, countable

$F_3 = (\frac{1}{3} + \mathbb{Z}) \cup (\frac{2}{3} + \mathbb{Z})$, countable

$F_4 = (\frac{1}{4} + \mathbb{Z}) \cup (\frac{3}{4} + \mathbb{Z})$, countable

$F_5 = (\frac{1}{5} + \mathbb{Z}) \cup (\frac{2}{5} + \mathbb{Z}) \cup \cdots (\frac{4}{5} + \mathbb{Z})$, countable

$F_n = \cup (\frac{a}{n} + \mathbb{Z})$, $a < n$, $\gcd(a,n) = 1$, countable.